Introduction to BRST symmetry and Slavnov-Taylor identities

Christian Schwinn

Albert-Ludwigs-Universität Freiburg, Physikalisches Institut
D-79104 Freiburg, Germany

March 2, 2015

1 Gauge fixing and ghost Lagrangian

1.1 Classical QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{Q} \left(\mathrm{i} \not{D} - m_q \right) Q \tag{1}$$

with field-strength tensor

$$F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g_s f^{abc} A^b_\mu A^c_\nu \tag{2}$$

and the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i} g_s T^a A^a_\mu \tag{3}$$

Invariance under gauge transformations

$$\begin{aligned} \Delta Q(x) &= -\mathrm{i} g_s \omega^a(x) T^a Q(x) \\ \Delta A^a_\mu &= \partial_\mu \omega^a + g_s f^{abc} \omega^b A^c_\mu \end{aligned} \tag{4}$$

1.2 Gauge fixing and ghost Lagrangian

Covariant gauge fixing term

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 \tag{5}$$

The Fadeev-Popov Lagrangian is given by

$$\mathcal{L}_{\mathrm{FP}} = (\partial^\mu \bar{c}^a) D^{(\mathrm{ad})}_{ab,\mu} c^b = (\partial^\mu \bar{c}^a)(\partial_\mu \delta_{ab} + g_s f^{abc} A_{c,\mu}) c^b \tag{6}$$

with the *ghost fields* c^a and anti-ghost fields $\bar{c}^a$, scalar fields in the adjoint representation of $SU(3)$, which, however, are assigned Fermi-statistic. These fields never appear as external states, so the spin-statistics theorem is not violated by the wrong statistics.

The ghost fields are anticommuting scalars. This implies that

$$(\bar{c}^a c^b)^\dagger = c^{b\dagger}\bar{c}^{a\dagger} = -\bar{c}^{a\dagger}c^{b\dagger} \tag{7}$$

The Lagrangian is hermitian for the assignment

$$c^{a\dagger} = c^a \qquad\qquad \bar{c}^{a\dagger} = -\bar{c}^a \tag{8}$$

It is not consistent with a hermitian interaction term with the gauge boson to take the antighost as the conjugate of the ghost. The antighost could be made hermitian by a redefinition $\bar{c} \to \mathrm{i}\bar{c}$ but we will keep the form of the Lagrangian given above.

More generally, one can consider a gauge fixing term of the form

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(f^a[A_\mu])^2 \tag{9}$$

with some gauge-fixing functional $f^a[A_\mu]$. The Fadeev-Popov Lagrangian then involves the gauge variation of the gauge-fixing term:

$$\mathcal{L}_{\mathrm{FP}} = \int \mathrm{d}^4y\, \bar{c}^a(x)\frac{\delta f^a[A'_\mu(x)]}{\delta\omega^b(y)}c^b(y) = \bar{c}^a(x)\mathcal{M}^{ab}c^b(x) \tag{10}$$

where $A'_\mu = A_\mu + \Delta A_\mu$ and the last expression holds for a local gauge-fixing functional. For the covariant gauge fixing one obtains the previous result:

$$f^a = \partial_\mu A^{a,\mu} \qquad \Rightarrow \qquad \mathcal{M}^{ab} = \partial_\mu(\partial^\mu\delta^{ab} + g_s f^{abc}A^\mu_c) \tag{11}$$

1.3 Nakanishi-Lautrup auxiliary field

The gauge-fixing Lagrangian can be rewritten in terms of the so-called Nakanishi-Lautrup auxiliary fields B_a as

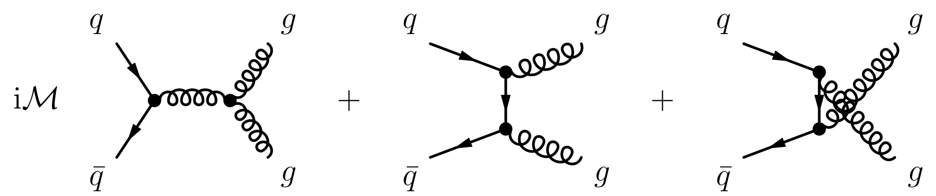
$$\mathcal{L}_{\mathrm{gf}} = B^a f^a + \frac{\xi}{2}(B^a)^2 \tag{12}$$

as can be seen using the equation of motion for the auxiliary fields

$$0 = \frac{\partial\mathcal{L}}{\partial B^a} = \xi B^a + f^a \tag{13}$$

1.4 Example for relevance of ghosts

Consider the matrix element for $q\bar{q} \to gg$



Define the “reduced” amplitude obtained by removing both of the gluon polarization vectors:

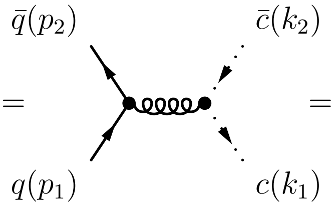
$$\mathcal{M} \;=\; \epsilon^{\mu *}(k_1)\epsilon^{\nu *}(k_2)\,\tilde{\mathcal{M}}_{\mu\nu} \tag{14}$$

One finds that the reduced amplitude satisfies the identity

$$\mathrm{i}\tilde{\mathcal{M}}_{\mu\nu}k_1^{\mu} = \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^{\mu}u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}(-g_s f^{abc}k_1^{\mu}k_{2,\nu}) \tag{15}$$

In QED the Ward identities imply that for the analogous photon amplitude the right-hand side vanishes. In non-abelian gauge theories the right-hand side is instead given by the matrix element for the (unphysical) “process” $q\bar{q} \to \bar{c}^a(k_1)c^b(k_2)$:

$$\mathrm{i}\mathcal{M}_{c_{k_1}\bar{c}_{k_2}} =$$



$$= \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^{\mu}u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}(-g_s f^{bac}k_1^{\mu}) \tag{16}$$

Therefore one finds that the violation of the WI for the reduced amplitude is proportional to the ghost diagram:

$$\tilde{\mathcal{M}}_{\mu\nu}k_1^{\mu} = -k_{2,\nu}\mathcal{M}_{c_{k_1}\bar{c}_{k_2}} \tag{17}$$

This identity can be derived from the so-called BRST invariance of the gauge-fixed Lagrangian.

2 BRST transformations

2.1 Definition of the BRST transformation

The gauge-fixed QCD Lagrangian including the Fadeev-Popov Lagrangian is invariant under a global transformation parametrized by a Grassmann-valued parameter θ. This transformation was discovered by Becchi, Rouet, Stora, and Tyutin (BRST). We write the transformations of a general field Ψ as

$$\Delta_\theta\Psi = \theta\delta_{\mathrm{B}}\Psi \tag{18}$$

The transformations of the physical fields are obtained from the infinitesimal gauge transformations by replacing the parameters ω^a by the product of the ghost fields and the Grassmann number θ:

$$\delta_{\mathrm{B}} A^a_\mu = \partial_\mu c^a + g_s f^{abc} c^b A^c_\mu \tag{19}$$

$$\delta_{\mathrm{B}} Q(x) = -\mathrm{i} g_s c^a(x) T^a Q(x) \tag{20}$$

The transformations of the ghost fields and the auxiliary fields are taken as

$$\delta_{\mathrm{B}} c^a = \frac{1}{2} g_s f^{abc} c^b c^c \tag{21}$$

$$\delta_{\mathrm{B}} \bar{c}^a = B^a \tag{22}$$

$$\delta_{\mathrm{B}} B^a = 0 \tag{23}$$

2.2 BRST charge

We introduce the generator of BRST transformations Q_{B}, the so-called BRST charge:

$$\Delta_\theta \Psi = \theta \delta_{\mathrm{B}} \Psi \equiv [\mathrm{i}\theta Q_{\mathrm{B}}, \Psi]. \tag{24}$$

The BRST charge can be constructed explicitly using the Noether theorem [2]. The BRST transformations of bosonic fields are generated by commutators with the BRST charge, the transformations of fermionic fields by anticommutators:

$$[Q_{\mathrm{B}}, \Phi]_\pm = -\mathrm{i}\delta_{\mathrm{B}} \Phi \tag{25}$$

With the above definition of the conjugations of the ghost and anti-ghost fields, the BRST charge is hermitian:

$$[Q_{\mathrm{B}}, A^a]^\dagger = \mathrm{i}(\partial_\mu c^a + g_s f^{abc} c^b A^c_\mu) = [A^a, Q_{\mathrm{B}}] \tag{26}$$

$$\{Q_{\mathrm{B}}, Q\}^\dagger = -g_s Q^\dagger T^a c^a = g_s c^a Q^\dagger T^a = \{Q^\dagger, Q_{\mathrm{B}}\} \tag{27}$$

$$\{Q_{\mathrm{B}}, \bar{c}^a\}^\dagger = \mathrm{i} B^a = \{\bar{c}^{a\dagger}, Q_{\mathrm{B}}\} \tag{28}$$

$$\{Q_{\mathrm{B}}, c^a\}^\dagger = \frac{\mathrm{i}}{2} g_s f^{abc} c^c c^b = -\frac{\mathrm{i}}{2} g_s f^{abc} c^b c^c = \{c^{a\dagger}, Q_{\mathrm{B}}\} \tag{29}$$

The BRST transformation of products of fields is defined as

$$\begin{aligned} \Delta_\theta(\Psi_1 \dots \Psi_n) &= [\mathrm{i}\theta Q_{\mathrm{B}}, \Psi_1 \dots \Psi_n] \\ &= \theta(\delta_{\mathrm{B}}\Psi_1) \dots \Psi_n + \dots \theta(-1)^{s_i} \Psi_1 \dots (\delta_{\mathrm{B}}\Psi_i) \dots \Psi_n \\ &\equiv \theta \delta_{\mathrm{B}}(\Psi_1 \dots \Psi_n) \end{aligned} \tag{30}$$

where s_i counts the number of fermionic fields before Ψ_i. The last line defines the action of δ_{B} on products of fields.

Note that we have

$$\begin{aligned} \delta^2_{\mathrm{B}}(\Psi_1\Psi_2) &= \delta_{\mathrm{B}}\left[(\delta_{\mathrm{B}}\Psi_1)\Psi_2 + (-1)^{s_1}\Psi_1(\delta_{\mathrm{B}}\Psi_2)\right] \\ &= (\delta^2_{\mathrm{B}}\Psi_1)\Psi_2 - (-1)^{s_1}(\delta_{\mathrm{B}}\Psi_1)(\delta_{\mathrm{B}}\Psi_2) + (-1)^{s_1}(\delta_{\mathrm{B}}\Psi_1(\delta_{\mathrm{B}}\Psi_2) + \Psi_1(\delta^2_{\mathrm{B}}\Psi_2) \\ &= (\delta^2_{\mathrm{B}}\Psi_1)\Psi_2 + \Psi_1(\delta^2_{\mathrm{B}}\Psi_2) \end{aligned} \tag{31}$$

2.3 Properties

The BRST transformation has the following properties:

1. it leaves the Lagrangian invariant
2. it is *nilpotent*, i.e. for any field one has
$$\delta_{\mathrm{B}}^2 \Phi = 0 \tag{32}$$
Because of (31) this implies automatically that $\delta_{\mathrm{B}}^2 F = 0$ for any functional F of the fields.
3. The sum of the gauge-fixing and ghost Lagrangians can be written as
$$\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{FP}} = \delta_{\mathrm{B}} \mathcal{F} \tag{33}$$
with
$$\mathcal{F} = \bar{c}^a (f^a + \frac{\xi}{2} B^a) \tag{34}$$
For the choice of the gauge fixing function $f^a = \partial_\mu A^{a,\mu}$ this is easily seen:
$$\delta_{\mathrm{B}} \mathcal{F} = B^a (\partial_\mu A^{a,\mu} + \frac{\xi}{2} B^a) - \bar{c}^a \partial_\mu (\partial^\mu c^a + g_s f^{abc} c^b A^{c,\mu}) \tag{35}$$
For a general gauge fixing functional one uses
$$\delta_{\mathrm{B}} f^a[A] = \int \mathrm{d}^4 y \frac{\delta f^a[\mathcal{A}'_\mu(x)]}{\delta \omega^b(y)} \theta c^b = \mathcal{M}^{ab} \theta c^b \tag{36}$$
so that
$$\delta_{\mathrm{B}} \mathcal{F} = B^a (f^a + \frac{\xi}{2} B^a) - \bar{c}^a \mathcal{M}^{ab} c^b \tag{37}$$

The BRST invariance of the Lagrangian follows from the two other properties and the fact that the classical QCD Lagrangian is invariant by construction.

2.3.1 Proof of nilpotency

It remains to be shown that the BRST transformation is nilpotent.

The nilpotency is obvious for the antighost and the auxiliary field (in fact, this is the motivation for introducing the auxiliary field, since the nilpotency holds only after use of the equations of motion if the formulation without auxiliary fields is used).

For the quark field we have
$$\begin{aligned}
\delta_{\mathrm{B}}^2 Q &= -\mathrm{i} g_s \left((\delta_{\mathrm{B}} c^a) T^a Q - c^a T^a \delta_{\mathrm{B}} Q \right) \\
&= -\mathrm{i} g_s^2 \Big(\frac{1}{2} c^b c^c f^{abc} T^a Q + \mathrm{i} \underbrace{c^b c^c T^b T^c}_{\frac{1}{2} c^b c^c [T^b, T^c]} Q \Big) \\
&= 0
\end{aligned} \tag{38}$$

For the gluon field, the repeated application of the BRS transformation gives

$$\begin{aligned}\delta_{\mathrm{B}}^2 A_\mu^a &= \partial_\mu \delta_{\mathrm{B}} c^a + g_s f^{abc}(\delta_{\mathrm{B}} c^b) A_\mu^c - g_s f^{abc} c^b \delta_{\mathrm{B}} A_\mu^c \\ &= \frac{g_s}{2}\left[f^{abc}\partial_\mu(c^b c^c) + g_s f^{abc} f^{bde} c^d c^e A_\mu^c - 2 f^{abc} c^b \left(\partial_\mu c^c + g_s f^{cde} c^d A_\mu^e\right)\right]\end{aligned} \tag{39}$$

This vanishes as can be seen separately for the derivative terms and the terms with the gauge fields, using the anticommuting nature of the ghosts

$$f^{abc}\left(\partial_\mu(c^b c^c) - 2c^b \partial_\mu c^c\right) = f^{abc}\left(\partial_\mu(c^b c^c) - c^b(\partial_\mu c^c) - (\partial_\mu c^b) c^c\right) \qquad = 0 \tag{40}$$

$$\begin{aligned}f^{abc} f^{bde} c^d c^e A_\mu^c - 2 f^{abc} f^{cde} c^b c^d A_\mu^e &= A_\mu^c \left(f^{abc} f^{bde} c^d c^e + 2 f^{abe} f^{edc} c^d c^b\right) \\ &= A_\mu^c c^d c^e \left(f^{abc} f^{bde} + 2 f^{aeb} f^{bdc}\right) \\ &= A_\mu^c c^d c^e \left(f^{abc} f^{bde} + f^{aeb} f^{bdc} - f^{adb} f^{bec}\right) \qquad = 0\end{aligned} \tag{41}$$

In the last line the Jacobi identity

$$f^{bcd} f^{ade} + f^{abd} f^{cde} + f^{cad} f^{bde} = 0. \tag{42}$$

was used. Since the term $\sim cA$ in the transformation law of the gauge field is the same as for a matter field in the adjoint representation, the cancellations in this case work in the same way as for the quark term, up to replacing the generators in the fundamental by those in the adjoint representation.

The last step is the proof of nilpotency for the transformation of the ghost fields:

$$\begin{aligned}\delta_{\mathrm{B}}^2 c^a &= \frac{1}{2} g_s f^{abc}\left[(\delta_{\mathrm{B}} c^b) c^c - c^b(\delta_{\mathrm{B}} c^c)\right] \\ &= \frac{1}{4} g_s^2 f^{abc}\left[f^{bde} c^d c^e c^c - f^{cde} c^b c^d c^e\right] \\ &= \frac{1}{2} g_s^2 f^{abc} f^{bde} c^c c^d c^e = 0\end{aligned} \tag{43}$$

Since the product of three ghost fields does not change sign under cyclic permutations, this expression vanishes as a result of the Jacobi identity.

3 BRST symmetry and states in a gauge theory

The vector space of states of a gauge theory contains four modes of the gauge field and the ghosts and antighosts, whereas the classification of states using the representations of the Poincaré group shows that only two transverse polarizations of the vector fields should appear. The BRST symmetry allows to define "physical" states consistently and allows to show that the unphysical states decouple.

3.1 Physical states

The nilpotency of the BRST transformation implies also

$$Q_{\mathrm{B}}^2 = 0 \tag{44}$$

This can be seen by computing the double commutator, for instance for a bosonic field,

$$0 = \delta_{\mathrm{B}}^2 \Phi = \{\mathrm{i}Q_{\mathrm{B}}, [\mathrm{i}Q_{\mathrm{B}}, \Phi]\} = -(Q_{\mathrm{B}}^2\Phi - Q_{\mathrm{B}}\Phi Q_{\mathrm{B}} + Q_{\mathrm{B}}\Phi Q_{\mathrm{B}} + \Phi Q_{\mathrm{B}}^2) = -[Q_{\mathrm{B}}^2, \Phi] \tag{45}$$

This implies $Q_{\mathrm{B}}^2 = 0$. Note that the BRST transformation changes the ghost number by one, so that Q_{B}^2 must have ghost number two. This excludes the possibility that $Q_{\mathrm{B}}^2 \propto \mathbf{1}$.

Because of the nilpotency of Q, states that are obtained by applying Q_{B} to another arbitrary state (so called 'BRS exact states') have vanishing norm:

$$|\psi\rangle = Q_{\mathrm{B}} |\eta\rangle : \quad \langle\psi|\psi\rangle = 0 \qquad \forall\, |\eta\rangle \tag{46}$$

States that are annihilated by the BRS charge are called 'BRS closed'. They are orthogonal to the exact states:

$$\langle\psi|\phi\rangle = \langle\eta|Q_{\mathrm{B}}|\phi\rangle = 0 \qquad \forall\, |\psi\rangle = Q_{\mathrm{B}}|\eta\rangle \quad , \quad Q_{\mathrm{B}}|\phi\rangle = 0 \tag{47}$$

Therefore we can decompose the Hilbert space into orthogonal subspaces. Because of the nilpotency of Q_{B}, a closed state stays closed if one adds an arbitrary exact state.

One can show (see e.g. [2]) that provided the BRS closed states have positive norm, it is consistent to define the *physical* states of the theory as closed states modulo exact states:

$$\begin{aligned} Q_{\mathrm{B}} |\psi_{\text{phys}}\rangle &= 0 \\ |\psi_{\text{phys}}\rangle &\sim |\psi_{\text{phys}}\rangle + Q_{\mathrm{B}}|\eta\rangle \end{aligned} \tag{48}$$

In mathematical terms, this is the *cohomology* of the operator Q.

3.2 Asymptotic fields

Consider the asymptotic in/out states that satisfy the free equations of motion.

They admit the same mode decomposition as the free fields, i.e. for the gauge field (suppressing colour indices)

$$A^\mu(x) = \sum_{\lambda=\pm,,L,S} \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0} \left(a_\lambda(\vec{p})\epsilon^\mu_\lambda(p) e^{-ipx} + a^\dagger_\lambda(\vec{p})\epsilon^{\mu,*}_\lambda(p) e^{ipx} \right) \tag{49}$$

and the ghost fields

$$c(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0} \left(c(\vec{p}) e^{-ipx} + c^\dagger(\vec{p}) e^{ipx} \right) |_{p^0=|\vec{p}|} \tag{50}$$

$$\bar{c}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0} \left(\bar{c}(\vec{p}) e^{-ipx} - \bar{c}^\dagger(\vec{p}) e^{ipx} \right) |_{p^0=|\vec{p}|} \tag{51}$$

The sum over the polarization vectors is over the two transverse polarizations with polarization vectors $\epsilon_\pm$ and two additional "longitudinal" and "scalar" polarizations given by

$$\epsilon_L^\mu(\vec{p}) = \begin{pmatrix} |\vec{p}| \\ \vec{p} \end{pmatrix} = p^\mu \qquad \epsilon_S^\mu(\vec{p}) = \frac{1}{2|\vec{p}|^2} \begin{pmatrix} |\vec{p}| \\ -\vec{p} \end{pmatrix} \tag{52}$$

The polarization vectors are normalized as

$$(\epsilon_\lambda(p) \cdot \epsilon^*_{\lambda'}(p)) = -\delta_{\lambda\lambda'}, \qquad \lambda, \lambda' = \pm \tag{53}$$

$$(\epsilon_S(p) \cdot \epsilon_L(p)) = 1 \tag{54}$$

$$(\epsilon_{S/L}(p) \cdot \epsilon_{S/L}(p)) = 0 \tag{55}$$

The BRST transformations of asymptotic fields in Feynman gauge ($\xi = 1$) are obtained by the limit $g_s \to 0$:

$$[\mathrm{i}Q_\mathrm{B}, A^{a,\mu}(x)] = \partial_\mu c^a(x) \tag{56}$$

$$\{\mathrm{i}Q_\mathrm{B}, c^a(x)\} = 0 \tag{57}$$

$$\{\mathrm{i}Q_\mathrm{B}, \bar{c}^a(x)\} = B^a(x) = -\partial_\mu A^{a,\mu}(x) \tag{58}$$

$$[\mathrm{i}Q_\mathrm{B}, B^a(x)] = 0 \tag{59}$$

Inserting the mode decomposition and comparing coefficients, one finds the transformations of the creation operators

$$[Q_\mathrm{B}, a^\dagger_\pm(\vec{p})] = [Q_\mathrm{B}, a^\dagger_S(\vec{p})] = 0 \tag{60}$$

$$[Q_\mathrm{B}, a^\dagger_L(\vec{p})] = c^a(\vec{p}) \tag{61}$$

$$\{Q_\mathrm{B}, c^\dagger(\vec{p})\} = 0 \tag{62}$$

$$\{Q_\mathrm{B}, \bar{c}^\dagger(\vec{p})\} = -a^\dagger_S(\vec{p}) \tag{63}$$

$$[Q_\mathrm{B}, a^\dagger_S(x)] = 0 \tag{64}$$

Since the vacuum satisfies $Q_\mathrm{B} |0\rangle = 0$ the states obtained by acting with the creation operator on the vacuum are classified as follows:

- BRST exact (zero-norm) states:

$$|c\rangle = Q_\mathrm{B} |g, L\rangle \tag{65}$$

$$|g, S\rangle = -Q_\mathrm{B} |c\rangle \tag{66}$$

$$\tag{67}$$

- BRST closed but not exact states:

$$|g, \pm\rangle \tag{68}$$

- physical states: equivalence classes of closed modulo exact states:

$$|g, \pm\rangle \sim |g, \pm\rangle + \alpha\ |g, S\rangle \tag{69}$$

The physical Hilbert space $\mathcal{H}_{\rm phys}$ is usually defined as the equivalence class of the BRST-closed modulo exact states with ghost number zero. One can show [2] that because of the scalar products (54) and (55) the annihilation and creation operators of the unphysical modes satisfy the commutation relation

$$[a_S(\vec{k}), a_L^\dagger(\vec{p})] = \delta^3(\vec{k} - \vec{p}) \tag{70}$$

This implies that the S-matrix element for the state $|g, S\rangle$ is obtained by Feynman diagrams calculated with the polarization vector ϵ_L:

$$\langle \ldots A^\mu a_S^\dagger |0\rangle \to \langle \ldots |0\rangle\, \epsilon_L^\mu \tag{71}$$

Therefore the equivalence of the states (69) implies the equivalence of the polarization vectors

$$\epsilon_\pm^\mu \sim \epsilon_\pm^\mu + \alpha \epsilon_L^\mu = \epsilon_\pm^\mu + \alpha p^\mu, \tag{72}$$

i.e. the usual invariance under gauge transformations.

4 Consequences of BRST invariance

The BRST invariance is essential for the proofs of unitarity, gauge independence and renormalizability of gauge theories. To illustrate this, we briefly sketch how it can be used to show the gauge independence of S-matrix elements and to derive the Slavnov-Taylor identities.

4.1 Gauge independence

We have seen above that the gauge fixed Lagrangian has the form

$$\mathcal{L} = \mathcal{L}_0 + \delta_{\rm B}\mathcal{F} \tag{73}$$

with

$$\mathcal{F} = \bar{c}^a (f^a + \frac{\xi}{2} B^a) \tag{74}$$

Consider a variation of the gauge-fixing functional

$$f^a[A] \to f^a[A] + \Delta f^a[A], \tag{75}$$

which implies the variation of the gauge fixing Lagrangian

$$\delta\mathcal{L} = \delta_{\rm B}(\bar{c}^a \delta f^a) = \{{\rm i}Q_{\rm B}, \bar{c}^a \Delta f^a[A]\}. \tag{76}$$

One can show that the change of a matrix element under the variation of the gauge fixing term is given by

$$\begin{aligned}\langle\phi_{\rm phys}|\psi_{\rm phys}\rangle_{f+\Delta f} - \langle\phi_{\rm phys}|\psi_{\rm phys}\rangle_{f} &= \int d^4x\, \langle\phi_{\rm phys}|\mathrm{i}\delta\mathcal{L}(x)|\psi_{\rm phys}\rangle_f \\ &= \mathrm{i}\int d^4x\, \langle\phi_{\rm phys}|\{\mathrm{i}Q_{\rm B}, \bar{c}^a(x)\Delta f^a[A(x)]\}|\psi_{\rm phys}\rangle \\ &= 0,\end{aligned} \tag{77}$$

which vanishes because of the definition of the physical states. For a more careful discussion including the LSZ reduction and renormalization see [2, 3].

4.2 Slavnov Taylor identities

We can derive the general *Slavnov Taylor Identities* of the theory by sandwiching the commutator (or anticommutator) of an arbitrary products of fields with the BRS-charge between physical fields:

$$\begin{aligned}0 &= \langle\phi_{\rm phys}|\,\mathrm{T}[[\mathrm{i}Q_B, \Psi_1\Psi_2\ldots\Psi_n]_\pm]|\psi_{\rm phys}\rangle \\ &= \sum_i (-)^{s(i)}\,\langle\phi_{\rm phys}|\,\mathrm{T}[\Psi_1\ldots\delta_{\rm B}\Psi_i\ldots\Psi_n]|\psi_{\rm phys}\rangle\end{aligned} \tag{78}$$

As example consider identity obtained from the matrix element

$$\langle 0|\bar{c}^a(x)A^{\mu,b}(y)|Q,\bar{Q}\rangle_{\rm phys} \tag{79}$$

The STI implies

$$0 = \langle 0|(\delta_{\rm B}\bar{c}^a(x))A^{\mu,b}(y)|Q,\bar{Q}\rangle - \langle 0|\bar{c}^a(x)\delta_{\rm B}A^{\mu,b}(y)|Q,\bar{Q}\rangle \tag{80}$$

i.e.

$$\langle 0|\underbrace{B^a(x)}_{=-\partial_\mu A^a} A^{\mu,b}(y)|Q,\bar{Q}\rangle = \langle 0|\bar{c}^a(x)(\partial_\mu c^b + g_s f^{abc}c^b A^c_\mu)(y)|Q,\bar{Q}\rangle \tag{81}$$

At tree-level the bilinear term in the transformation of the gluon field does not contribute and one has the relation

$$\partial_{x,\mu}\,\langle 0|A^a_\mu(x)A^{\mu,b}(y)|Q,\bar{Q}\rangle = \partial_{y,\mu}\,\langle 0|\bar{c}^a(x)c^b(y)|Q\bar{Q}\rangle \tag{82}$$

Performing the LSZ reduction on the photon and ghost fields one obtains the identity (17)

$$\tilde{\mathcal{M}}_{\mu\nu}k_1^\mu = -k_{2,\nu}\mathcal{M}_{c_{k_1}\bar{c}_{k_2}} \tag{83}$$

found previously by an explicit calculation. Note that the ghost propagator connects anti-ghost and ghost fields so the LSZ reduction of an antighost field gives a ghost amplitude.

References

[1] S. Weinberg, "The quantum theory of fields. Vol. 2: Modern applications," Cambridge, UK: Univ. Pr. (1996).

[2] T. Kugo, "Eichtheorie," Springer (1997).

[3] M. Böhm, A. Denner and H. Joos, "Gauge theories of the strong and electroweak interaction," Teubner (2001).